

# GRADIENT ESTIMATE FOR THE POISSON EQUATION AND THE NON-HOMOGENEOUS HEAT EQUATION ON COMPACT RIEMANNIAN MANIFOLDS

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**ABSTRACT.** In this short note, we study the gradient estimate of positive solutions to Poisson equation and the non-homogeneous heat equation in a compact Riemannian manifold  $(M^n, g)$ . Our results extend the gradient estimate for positive harmonic functions and positive solutions to heat equations.

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## 1. INTRODUCTION

In the process of the study of the positive solutions to non-local non-homogeneous heat equation (see [2],[10], [11], and [12]) in a compact Riemannian manifold  $(M, g)$ :

$$u_t = \Delta u + \lambda(t)u + f(x, t),$$

with the initial data  $u(0, x) = u_0(x)$ , where  $\int_M u_0(x)^2 = 1$  and

$$\lambda(t) = \int_M |\nabla u|^2 - fu,$$

we find that it is interesting to study the gradient estimate for positive solutions to the elliptic equation

$$-\Delta u = A(x), \text{ in } M$$

and the heat equation

$$(\partial_t - \Delta)u = A(x, t), \text{ in } M \times (0, T).$$

We shall follow the ideas in [9] and [13], which uses tricks from the works of Cheng-Yau [17] on harmonic functions and Li-Yau [17] on heat equations. However, some new contributions have to be provided since they have treated different situation as ours (see [4],[7],[9] and [15]). It is also clear that our result can be extended to complete Riemannian manifolds. For more related works on complete manifolds, one may look at [7], [8], [6], and [14].

It is a surprise to us that there is few literature about the gradient estimate for positive solutions to Poisson equations and non-homogeneous heat equations. With this understanding and motivated from the work of L.Caffarelli and F.H.Lin [2] and our paper [9], we consider the gradient estimates for Poisson equation and non-homogeneous heat equation in a compact Riemannian manifold.

We shall always assume that  $Ric(g) \geq K$  on  $M$  for some real constant  $K$ .

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For the Poisson equation, we have the following result.

**Theorem 1.** *Let  $u > 0$  be a smooth solution to the Poisson equation on  $M$*

$$-\Delta u = A(x).$$

*Then we have*

$$|\nabla w|^2 + A(x)u^{-1} \leq 2n \sup\{K - Au^{-1}, A^2u^{-2} - [n^{-1}4(Au^{-1} - K)^2 + 2KAu^{-1} + u^{-1}\Delta A]\},$$

For the non-homogeneous heat equation on  $M \times [0, T)$ , we have the following result.

**Theorem 2.** *Let  $u > 0$  be a smooth solution to the non-homogeneous heat equation on  $M \times [0, T)$*

$$(\partial_t - \Delta)u = A(x, t).$$

*Let, for  $a > 1$ ,*

$$F = t(|\nabla w|^2 + aA(x, t)u^{-1} - aw_t).$$

*Then there is a constant  $C(u^{-1}, |A|, |\nabla A|, |\Delta A|, K, a, T) > 0$  such that*

$$\sup_{M \times (0, T)} F \leq C(u^{-1}, |A|, |\nabla A|, |\Delta A|, K, a, T).$$

Related local gradient estimates can be extended to complete non-compact Riemannian manifolds, which will appear elsewhere.

This paper is organized as follows. In section 2 we prove Theorem 1. In section 3 we do the gradient estimate for a positive smooth solution to the non-homogeneous heat equation.

## 2. GRADIENT ESTIMATE FOR POISSON EQUATION

We firstly prove Theorem 1 about the gradient estimate for parabolic equations. Recall here that we are considering the object for Poisson equation on the compact Riemannian manifold  $(M^n, g)$ .

We now recall the famous Bochner formula any smooth function  $v$  on a Riemannian manifold  $(M^n, g)$ :

$$(2.1) \quad \Delta|\nabla v|^2 = 2|D^2v|^2 + 2(\nabla v, \nabla \Delta v) + 2Ric(\nabla v, \nabla v).$$

Recall that  $|D^2v|^2 \geq \frac{1}{n}|\Delta v|^2$ . So we have

$$\Delta|\nabla v|^2 \geq \frac{2}{n}|\Delta v|^2 + 2(\nabla v, \nabla \Delta v) + 2Ric(\nabla v, \nabla v)$$

This formula will plays a key role in our gradient estimate.

Let  $u > 0$  be a smooth solution to the Poisson equation on  $M$

$$-\Delta u = A(x).$$

Set

$$w = \log u.$$

Then we have

$$(2.2) \quad -\Delta w = |\nabla w|^2 + A(x)u^{-1}.$$

Let  $Q = |\nabla w|^2 + A(x)u^{-1}$  be the Harnack quantity. Then  $Q = -\Delta w$ .

By (3.2), we obtain that

$$\Delta Q = \Delta|\nabla w|^2 + \Delta(A(x)u^{-1}).$$

Using the Bochner formula (2.1), we get

$$\Delta|\nabla w|^2 \geq \frac{2}{n}Q^2 + 2(\nabla v, \nabla Q) - 2K|\nabla w|^2.$$

Then we have

$$\Delta Q \geq \frac{2}{n}Q^2 + 2(\nabla w, \nabla Q) - 2K(Q - A(x)u^{-1}) + \Delta(A(x)u^{-1})$$

Note that

$$\Delta(A(x)u^{-1}) = u^{-1}\Delta A(x) - 2u^{-1}\nabla A \cdot \nabla w + Au^{-1}(2Q - Au^{-1}).$$

Then at the maximum point  $p \in M$  of  $Q$  (which can be assumed positive), we have

$$\Delta Q \leq 0, \quad \nabla Q = 0.$$

Then we have

$$0 \geq \frac{2}{n}Q^2 + (2Au^{-1} - 2K)Q + 2KAu^{-1} + u^{-1}\Delta A - A^2u^{-2} - 2u^{-1}\nabla A \cdot \nabla w.$$

Using the Cauchy-Schwartz inequality we get for any  $b > 0$ ,

$$\frac{2}{n}[Q + n(Au^{-1} - b - K)]^2 + n^{-1}4(Au^{-1} - b - K)^2 + 2KAu^{-1} + u^{-2}(u\Delta A - \frac{|\nabla A|^2}{2b} - A^2) \leq 0.$$

If  $Q > 2n(K + b - Au^{-1})$ , then

$$Q + n(Au^{-1} - b - K) > Q/2 > 0.$$

Hence we have

$$\frac{1}{2n}Q^2 \leq (A^2 + \frac{|\nabla A|^2}{2b})u^{-2} - [n^{-1}4(Au^{-1} - K)^2 + 2KAu^{-1} + u^{-1}\Delta A].$$

In conclusion we have

$$Q \leq 2n \sup\{K + b - Au^{-1}, (A^2 + \frac{|\nabla A|^2}{2b})u^{-2} - [n^{-1}4(Au^{-1} - K)^2 + 2KAu^{-1} + u^{-1}\Delta A]\}$$

This implies that by choosing  $b = 1/2$ ,

$$|\nabla w|^2 + A(x)u^{-1} \leq 2n \sup\{K + \frac{1}{2} - Au^{-1}, (A^2 + |\nabla A|^2)u^{-2} - [n^{-1}4(Au^{-1} - K)^2 + u^{-1}(2KA + \Delta A)]\},$$

which is the gradient estimate wanted for positive solutions to the Poisson equation.

This completes the proof of Theorem 1.

### 3. GRADIENT ESTIMATE FOR NON-HOMOGENEOUS HEAT EQUATION

We now prove Theorem 2. Let  $u > 0$  be a smooth solution to the non-homogeneous heat equation on  $M \times [0, T)$

$$(3.1) \quad (\partial_t - \Delta)u = A(x, t).$$

Set

$$w = \log u.$$

Then we have

$$(3.2) \quad (\partial_t - \Delta)w = |\nabla w|^2 + Au^{-1}.$$

Following Li-Yau [17] we let  $F = t(|\nabla w|^2 + aAu^{-1} - aw_t)$  (where  $a > 1$ ) be the Harnack quantity for (3.1). Then we have

$$|\nabla w|^2 = \frac{F}{t} - aAu^{-1} + aw_t,$$

$$\Delta w = w_t - |\nabla w|^2 - Au^{-1} = -\frac{F}{at} - (1 - \frac{1}{a})|\nabla w|^2.$$

and

$$w_t - \Delta w = |\nabla w|^2 + Au^{-1} = \frac{F}{t} + (1 - a)Au^{-1} + aw_t.$$

Note that

$$(\partial_t - \Delta)w_t = 2\nabla w \nabla w_t + \frac{d}{dt}(Au^{-1}).$$

Using the Bochner formula, we have

$$(\partial_t - \Delta)|\nabla w|^2 = 2\nabla w \nabla w_t - [2|D^2w|^2 + 2(\nabla w, \nabla \Delta w) + 2Ric(\nabla w, \nabla w)],$$

and using (3.2) we get

$$(\partial_t - \Delta)|\nabla w|^2 = 2\nabla w \nabla(w_t - \Delta w) - [2|D^2w|^2 + 2Ric(\nabla w, \nabla w)],$$

which can be rewritten as

$$(\partial_t - \Delta)|\nabla w|^2 = 2\nabla w \nabla[\frac{F}{t} + (1 - a)Au^{-1} + aw_t] - [2|D^2w|^2 + 2Ric(\nabla w, \nabla w)].$$

Then we have

$$\begin{aligned} (\partial_t - \Delta)(|\nabla w|^2 - aw_t) &= 2\nabla w \nabla[\frac{F}{t} + (1 - a)Au^{-1}] \\ &\quad - [2|D^2w|^2 + 2Ric(\nabla w, \nabla w)] - a\frac{d}{dt}(Au^{-1}). \end{aligned}$$

Hence

$$\begin{aligned} &(\partial_t - \Delta)(|\nabla w|^2 - aw_t + aAu^{-1}) \\ &= (\partial_t - \Delta)(|\nabla w|^2 - aw_t) + a(\partial_t - \Delta)Au^{-1} \\ &= 2\nabla w \nabla[\frac{F}{t} + (1 - a)Au^{-1}] - [2|D^2w|^2 + 2Ric(\nabla w, \nabla w)] - a\frac{d}{dt}(Au^{-1}) \\ &\quad + a(\partial_t - \Delta)Au^{-1} \\ &= 2\nabla w \nabla[\frac{F}{t} + (1 - a)(Au^{-1})] - [2|D^2w|^2 + 2Ric(\nabla w, \nabla w)] - a\Delta(Au^{-1}). \end{aligned}$$

Then we have

$$\begin{aligned} (\partial_t - \Delta)F &= \frac{F}{t} + 2t\nabla w \nabla[\frac{F}{t} + (1 - a)(Au^{-1})] \\ &\quad - t[2|D^2w|^2 + 2Ric(\nabla w, \nabla w)] - at\Delta(Au^{-1}). \end{aligned}$$

Assume that

$$\sup_{M \times [0, T]} F > 0.$$

Applying the maximum principle at the maximum point  $(z, s)$ , we then have

$$(\partial_t - \Delta)F \geq 0, \quad \nabla F = 0.$$

In the following our computation is always at the point  $(z, s)$ . So we get

$$(3.3) \quad \frac{F}{s} + 2(1 - a)s\nabla w \nabla(Au^{-1}) - s[2|D^2w|^2 + 2Ric(\nabla w, \nabla w)] - as\Delta(Au^{-1}) \geq 0.$$

That is

$$(3.4) \quad F - as^2\Delta(Au^{-1}) \geq 2(a - 1)s^2\nabla w \nabla(Au^{-1}) + s^2[2|D^2w|^2 + 2Ric(\nabla w, \nabla w)].$$

Set

$$\mu = \frac{|\nabla w|^2}{F}(z, s).$$

Then at  $(z, s)$ ,

$$|\nabla w|^2 = \mu F.$$

Hence

$$\nabla \frac{A}{u} = \frac{\nabla A}{u} - \frac{A \nabla u}{u^2} = \frac{\nabla A}{u} - \frac{A}{u} \nabla w.$$

So

$$\begin{aligned} \nabla w \cdot \nabla \frac{A}{u} &= \frac{\nabla w \cdot \nabla A}{u} - \frac{A}{u} |\nabla w|^2 \geq -\frac{|\nabla w| |\nabla A|}{u} - \frac{A}{u} |\nabla w|^2 \\ (3.5) \quad &= -\frac{|\nabla A|}{u} \sqrt{\mu F} - \frac{A}{u} \mu F \geq -\frac{1}{2} \frac{|\nabla A|^2}{u} - \left(\frac{1}{2} + A\right) \frac{\mu F}{u}. \end{aligned}$$

Further more, we have

$$\begin{aligned} (\partial_t - \Delta)(Au^{-1}) &= \frac{1}{u} (\partial_t - \Delta)A - \frac{A}{u^2} (\partial_t - \Delta)u + \frac{2}{u^2} \nabla u \cdot \nabla A - 2 \frac{A}{u^3} |\nabla u|^2 \\ &= \frac{1}{u} (\partial_t - \Delta)A - \frac{A^2}{u^2} + \frac{2}{u} \nabla w \cdot \nabla A - 2 \frac{A}{u} |\nabla w|^2 \\ &\leq \frac{1}{u} (\partial_t - \Delta)A - \frac{A^2}{u^2} + \frac{2}{u} \sqrt{\mu F} |\nabla A| - 2 \frac{A}{u} \mu F \\ &\leq \frac{1}{u} (\partial_t - \Delta)A - \frac{A^2}{u^2} + \frac{\mu F}{u} + \frac{|\nabla A|^2}{u} - 2 \frac{A}{u} \mu F, \end{aligned}$$

and

$$\begin{aligned} \partial_t(Au^{-1}) &= \frac{A_t}{u} - \frac{A}{u^2} u_t \\ &= \frac{A_t}{u} - \frac{A}{u} w_t \\ &= \frac{A_t}{u} - \frac{A}{u} \left( \frac{1}{a} (|\nabla w|^2 - \frac{F}{s}) + Au^{-1} \right) \\ &= \frac{A_t}{u} - \frac{A}{u} \cdot \frac{F}{a} \left( \mu - \frac{1}{s} \right) - \frac{A^2}{u^2}. \end{aligned}$$

Hence

$$\begin{aligned} (3.6) \quad -\Delta(Au^{-1}) &= (\partial_t - \Delta)(Au^{-1}) - \partial_t(Au^{-1}) \\ &\leq -\frac{\Delta A}{u} + \frac{|\nabla A|^2}{u} + \frac{1-2A}{u} \mu F + \frac{A}{u} \cdot \frac{F}{a} \left( \mu - \frac{1}{s} \right) \\ &< -\frac{\Delta A}{u} + \frac{|\nabla A|^2}{u} + \frac{1-2A}{u} \mu F + \frac{A}{u} \cdot \frac{F}{a} \mu. \end{aligned}$$

Note that

$$|D^2 w|^2 + Ric(\nabla w, \nabla w) \geq \frac{1}{n} |\Delta w|^2 - K |\nabla w|^2.$$

So

$$\begin{aligned} |D^2 w|^2 + Ric(\nabla w, \nabla w) &\geq \frac{1}{n} \left( \frac{F}{as} + \left(1 - \frac{1}{a}\right) |\nabla w|^2 \right)^2 - K |\nabla w|^2 \\ (3.7) \quad &= \frac{F^2}{n} \left( \frac{1}{as} + \left(1 - \frac{1}{a}\right) \mu \right)^2 - K \mu F. \end{aligned}$$

Substitute (3.5) (3.6) and (3.7) into (3.4), we get

$$F + \frac{as^2}{u} (-\Delta A + |\nabla A|^2) + \mu F \frac{s^2}{u} (a + (1-2a)A)$$

$$\geq -s^2(a-1)\frac{|\nabla A|^2}{u} - (a-1)s^2\frac{1+2A}{u}\mu F + \frac{2F^2}{n}\left(\frac{1}{a} + \left(1 - \frac{1}{a}\right)\mu s\right)^2 - 2s^2K\mu F.$$

Assume that

$$F \geq \frac{as^2}{u}(-\Delta A + |\nabla A|^2) + s^2(a-1)\frac{|\nabla A|^2}{u},$$

for otherwise we are done. Then we have

$$\begin{aligned} 2F + \mu F \frac{s^2}{u}(a + (1-2a)A) + (a-1)s^2\frac{1+2A}{u}\mu F + 2s^2K\mu F \\ \geq \frac{2F^2}{n}\left(\frac{1}{a} + \left(1 - \frac{1}{a}\right)\mu s\right)^2. \end{aligned}$$

Simplify this inequality, we get

$$\begin{aligned} \frac{2F}{n} \frac{1}{a^2} \leq \frac{2}{(1 + (a-1)\mu s)^2} + \frac{\mu s}{(1 + (a-1)\mu s)^2} \\ \cdot s(u^{-1}(a + (1-2a)A) + u^{-1}(a-1)(1+2A) + 2K). \end{aligned}$$

Hence we have the estimate for  $F$  at  $(z, s)$  such that

$$F(z, s) \leq C(u^{-1}, |A|, |\nabla A|, |\Delta A|, K, a, T),$$

which is the desired gradient estimate. This completes the proof of Theorem 2.

#### REFERENCES

- [1] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Monogr. Math., Springer-Verlag, Berlin, 1998.
- [2] C.Caffarelli, F.Lin, *Nonlocal heat flows preserving the  $L^2$  energy*, Discrete and continuous dynamical systems. 23, 49-64 (2009).
- [3] K.C.Chang, W.Y.Ding, and R.Ye, *Finite time blow-up of the heat flow of harmonic maps from surfaces*, JDG, 36(1992)507-515.
- [4] B.Chow, P.Lu, L.Ni. *Hamilton's Ricci Flow*. Science Press. American Mathematical Society, Beijing.Providence(2006).
- [5] Ben Chow and Richard Hamilton, *Constrained and linear Harnack inequalities for parabolic equations*, Inventiones Mathematicae 129, 213-238 (1997).
- [6] Xianzhe Dai and Li Ma, *Mass under Ricci flow*, Commun. Math. Phys., 274, 65-80 (2007).
- [7] L.Evans, *Partial Differential Equations*, Graduate studies in Math., AMS, 1986
- [8] R.Hamilton, *The formation of Singularities in the Ricci flow*, Surveys in Diff. Geom., Vol.2, pp7-136, 1995.
- [9] L.Ma, *Gradient estimates for a simple elliptic equation on complete non-compact Riemannian manifolds*, Journal of Functional Analysis, 241(2006)374-382.
- [10] L.Ma, A.Q.Zhu, *On a length preserving curve flow*. Preprint,2008.
- [11] L.Ma, L.Cheng, *A non-local area preserving curve flow*, Preprint, 2008.
- [12] L.Ma, L.Cheng, *non-local heat flows and gradient estimates on closed manifolds*, Preprint, 2009.
- [13] Li Ma, Baiyu, Liu, *Convex eigenfunction of a drifting Laplacian operator and the fundamental gap*, Pacific Journal of Math., 240(2009)343-361
- [14] L.Ma, Chong Li, and Lin Zhao, *Monotone solutions to a class of elliptic and diffusion equations*, CPAA, 6(2007)237-246.
- [15] Grisha Perelman, *The entropy formula for the Ricci flow and its geometric applications*,math.DG/0211159,2002.
- [16] R.Schoen, *Analytic aspects for Harmonic maps*, Seminar in PDE, edited by S.S.Chern, Springer, 1984.
- [17] R.Schoen and S.T.Yau, *Lectures on Differential Geometry*, international Press, 1994.

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